研 究

Analytical Design of Holographic Optical Scanning Elements

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A new design method for holographic optical scanning elements is described. The analytical expressions of their phase functions are derived through the relations between the mathematical expressions of an image point and the aberrations. For each of the translating and rotating scanning elements, a phase function is assumed as a power series of coordinates of points on the hologram, and its coefficients are determined so that ray aberrations may be kept small during scanning. The ray aberration functions are obtained through the analytical ray tracing, and this method can yield analytical phase functions.

1. Introduction

Computer-generated holograms may be used as holographic optical scanning elements. In a system consisting of such a hologram and a laser beam incident on its hologram, as the hologram moves, the focusing point of the diffracted beam moves on a prescribed curve. Scanners using computer-generated holograms were discussed by Bryngdahl and Lee.[1] Iwaoka and Shiozawa[2] numerically designed a linear holographic scanner by minimization of an inverse merit function. Ono and Nishida[3] proposed the method of semianalytical design by optimization of a phase transfer function. Herzig and Dändliker[4,5] published two papers on a differential method of designing holographic scanning elements, in the first[4] the second-order approximation was discussed and in the second[5] the extension to the higher order was studied.

In this paper, we propose an analytical design method for holographic optical elements for translating and rotating scanning. For analytical design of holographic optical elements, Winick and Fienup[6] attempted design of Fourier transform lens. They used the minimization of wavefront deviation. We intend to reduce the aberrations in Herzig and Dändliker's first paper. In our method, a phase function is assumed as a power series of coordinates of points on the hologram, and aberration functions are derived through the analytical ray tracing. The analytical phase functions are determined so that the aberrations may be kept small during scanning. We find that analytical phase functions which are proved to be significant improved approximation to optimum optical elements through simulation of their spot diagrams.

The notations and parameter settings in this paper frequently follow those given in Herzig and Dändliker's paper ; however, some of the sign conventions are different.

2. Aberration Function

We treat a plane transmission holographic optical element. The transmittance of the optical element is given by

$$T(x, y)=(1/2)+(1/2)\cos[\psi(x, y)], \quad (1)$$

where λ is the used wavelength of the light, and $\psi(x, y)$ denotes the phase function of the optical element. When only one pair of phase distributions of an input and its desired output wavefronts ψ_{in} and ψ_{out} on a hologram are given, we can obtain the trivial phase function ψ of the optical element, namely,

$$\psi=\psi_{in}-\psi_{out}. \quad (2)$$

However, in the case of scanning elements, during scanning we have a set of continuously different input phase distributions and another set of their corresponding desired output ones. It would be generally impossible to obtain an ideal optical element which converts every input wavefront in the set into each desired output wavefront. So the design is to determine parameters of an optimal optical element which make the differences between the emergent and desired wavefronts as small as possible over the set of inputs.

For this design, we derive analytical expressions for coordinates of an arriving point of a diffracted ray in an image plane in terms of the parameters of the optical element and the input data.

First, we take a Cartesian coordinate system whose xy plane is a hologram plane. We assume that the phase function of the optical element is represented by a power series of x and y as

$$\phi=(2\pi/\lambda)\sum_{i,j} a_{ij}x^i y^j. \quad (3)$$

We take the coefficients a_{ij} as the parameters by which the optical element is characterized. We denote direction cosines of an incident ray and its diffracted ray by l_0, m_0, n_0, and l, m, n, respectively. Then, the familiar grating equations hold for the incident ray impinging on the optical element, that is, for the first-order diffracted ray,

$$l=l_0+\frac{\lambda}{2\pi}\frac{\partial\phi}{\partial x},$$
$$m=m_0+\frac{\lambda}{2\pi}\frac{\partial\phi}{\partial y},$$
$$n=\sqrt{1-l^2-m^2}, \quad (4)$$

where the values of $\partial\phi/\partial x$ and $\partial\phi/\partial y$ are taken at the incident point. The diffracted ray with these direction cosines l, m, n proceeds to an image plane. Let the image plane be represented by

$$Ax+By+Cz=f, \quad (5)$$

where f is the distance between the optical element and the image plane. Then, the coordinates of the intersection point of the diffracted ray with the image plane, i. e., the spot coordinates, are given by

$$x'=x+kl,$$
$$y'=y+km,$$
$$z'=kn, \quad (6)$$

where x and y are the coordinates of the incident point on the optical element and k is given by

$$k=[f-(Ax+By)]/(Cn). \quad (7)$$

By using Eqs. (3)-(7), we can obtain the coordinates of the arriving point of the diffracted ray in the image plane. Since the imaging is aberration-free, every ray incident on an extended area of the optical element must converge to a single point on the image plane, the image point is independent of the incident point, and therefore the terms in the expressions of x' and y', which contain the coordinates of the incident point, represent the aberrations.

3. Translating Scanning Element

We consider the case in which the movement of a hologram is translation in its plane. In **Fig. 1**, a coordinate system (u, v, w) is fixed in space, and (x, y) is attached to the hologram plane and moves with the hologram. The hologram plane lies in the uv plane. A laser source is located at $(0, \rho\sin\gamma, \rho\cos\gamma)$. The sign of ρ is negative or positive according to divergence or convergence of the incident wavefront. We note that the sign convention for γ is not in agreement with that of Herzig and Dändliker.

As the hologram translates in the direction of the u axis, the focusing point of the diffracted rays generates a straight line parallel to the u axis on the plane $w=f$. Initially, the center of the hologram coincides with the origin of the (u, v, w) system, and the image is at $(0, 0, f)$. A phase function is assumed to be represented by Eq. (3). If a ray impinges initially at (x, y) on the hologram, then its coordinates coincide with (u, v). After displacement of the hologram by X, its coordinates of the incident point become

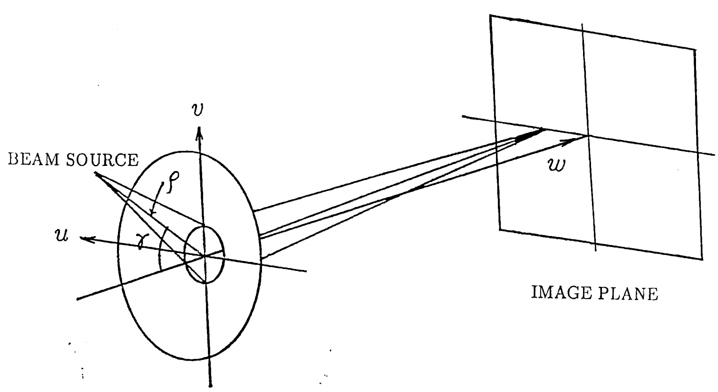

Fig. 1 Translating scanning element. The distance between the hologram plane and the image plane is f.

$$x=u+X,$$
$$y=v. \quad (8)$$

The direction cosines of an incident ray incident on (u, v) are represented by

$$l_0=u/k_0,$$
$$m_0=(v-\rho\sin\gamma)/k_0,$$
$$n_0=\rho\cos\gamma/k_0, \quad (9)$$

where

$$k_0=\sqrt{u^2+(v-\rho\sin\gamma)^2+(\rho\cos\gamma)^2}. \quad (10)$$

When utilizing Eqs. (4), we have to use $\partial\psi/\partial u$ and $\partial\psi/\partial v$ instead of $\partial\psi/\partial x$ and $\partial\psi/\partial y$, but we have the relations,

$$\frac{\partial\psi}{\partial x}=\frac{\partial\psi}{\partial u},$$
$$\frac{\partial\psi}{\partial y}=\frac{\partial\psi}{\partial v}. \quad (11)$$

By using Eq. (6), we can obtain coordinates u' and v' of an arriving point in the image plane as power series of u and v. In the expressions of u' and v', the terms containing u and v represent aberrations. We determine the coefficients a_{ij} by making their aberrations minima.

Let us determine the coefficients a_{ij} from the lowest order. First, at the initial position of the hologram, the principal diffracted ray proceeds along the w axis. From the fact that when $X=0$ then $l=0$, we obtain

$$a_{10}=0. \quad (12)$$

Since the generated curve is a straight line with $v=0$, m is also 0; therefore we obtain

$$a_{01}=\sin\gamma,\ a_{11}=0,\ a_{21}=0,\ a_{31}=0. \quad (13)$$

Hereafter, in the expressions of u' and v', we denote the terms of the ith order of u and v by u_i' and v_i', respectively. The terms which do not contain u and v are given by

$$u_0'=Xf(2a_{20}+4a_{20}{}^3X^2+3a_{30}X+4a_{40}X^2),$$
$$v_0'=0. \quad (14)$$

The term of the first order of X represents the linear displacement of image, while the second- and the higher-order terms represent nonlinear scanning.

The first-order terms of u and v represent astigmatism, the second-order terms coma, and the third-order terms spherical aberration, respectively. Consequently, through the appropriate choosing of a_{ij} in the phase function, we make the aberration terms zero or minimize them from the first-order terms.

The first-order terms of u and v are

$$u_1'=[2f\{a_{20}+(f+\rho)/(2f\rho)\}+6fa_{30}X$$
$$+12f\{a_{40}+(2\rho a_{20}{}^3+a_{20}{}^2)/(2\rho)\}X^2]u,$$
$$v_1'=[2f\{a_{02}+(f\cos^2\gamma+\rho)/(2f\rho)\}$$
$$+2fa_{12}X+2f\{a_{22}+(2a_{02}a_{20}{}^2\rho$$
$$+a_{20}{}^2\cos^2\gamma)/\rho\}X^2]v. \quad (15)$$

These astigmatisms vanish upon the choosing of coefficients as follows:

$$a_{20}=-(1/2)(1/f+1/\rho)$$
$$a_{02}=-(1/2)(1/f+\cos^2\gamma/\rho)$$
$$a_{30}=0$$
$$a_{12}=0$$
$$a_{40}=(1/8f)(1/f+1/\rho)^2$$
$$a_{22}=(1/4f)(1/f+1/\rho)^2. \quad (16)$$

The second-order terms of u and v are given by

$$u_2'=[3X(f+\rho)/(2f\rho^2)]u^2+[f\sin\gamma/\rho^2]uv$$
$$+[X(f+\rho)/(2f\rho^2)]v^2$$
$$v_2'=[f\sin\gamma/(2\rho^2)]u^2+[X(f+\rho)/(f\rho^2)]uv$$
$$+[3f(\cos^2\gamma\sin\gamma+2a_{03}\rho^2$$
$$+2a_{13}\rho^2X)]/(2\rho^2)]v^2. \quad (17)$$

Furthermore, for the third-order terms we find

$$u_3'=[(-f^2+f\rho+2\rho^2)/(2f\rho^3)]u^3$$
$$+[(-3f^2\cos^2\gamma+2f^2+f\rho$$
$$+2\rho^2)/(2f\rho^3)]uv^2+a_{13}fv^3$$
$$v_3'=[(-3f^2\cos^2\gamma+2f^2+f\rho+2\rho^2)/(2f\rho^3)]u^2v$$
$$+3a_{13}fuv^2+[(-5f^3\cos^4\gamma+4f^3\cos^2\gamma$$
$$+8a_{04}f^3\rho^3-\rho^3)/(2f^2\rho^3)]v^3. \quad (18)$$

The second- and third-order terms can not completely vanish; however, they may be minimized by the choosing of the coefficients a_{ij}. Letting the coefficient of v^2 in v_2' and the coefficient of v^3 in v_3' be zero, we obtain them as follows:

$$a_{03}=-(\cos^2\gamma\sin\gamma)/(2\rho^2),$$
$$a_{13}=-0,$$
$$a_{04}=(1/8)[(1/f^3+(5\cos^4\gamma-4\cos^2\gamma)/\rho^3]. \quad (19)$$

Thus, for the translating scanning element, we have obtained the coefficients of the phase function a_{ij} up to the fourth order from Eqs. (12), (13), (16), and (19).

4. Rotating Scanning Element

We consider the case in which the movement of a hologram is rotation in its plane. As in sec. 3, a hologram lies in the uv plane and rotates in its plane around $(0, -R, 0)$ during scanning, as shown in **Fig. 2**. A laser source is located at $(0, \rho\sin\gamma, \rho\cos\gamma)$. Unlike that used by Herzig and Dändliker, the sign of γ is positive when the source is in the positive v side, as in sec. 3. We assume that the focusing point of the diffracted rays generates a straight line parallel to u axis in the plane $w=f$, and that its distance from the uw plane is $\sin\alpha$.

In this case, as in Herzig and Dändliker, we represent the phase function by

$$\psi=(2\pi/\lambda)\sum_{ij}b_{ij}x^i(y-R)^j. \quad (20)$$

At the initial position, v and y axes are assumed

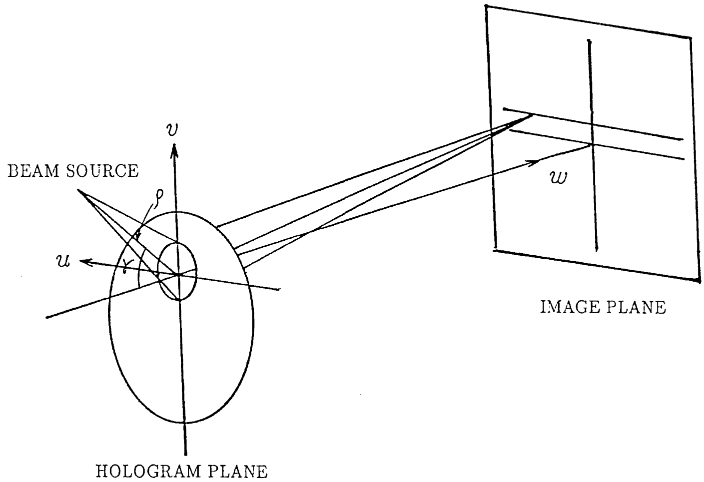

Fig. 2 Rotating scanning element. The distance between the hologram plane and the image plane is f, an in **Fig. 1**. The distance of the scanning line from uw plane is $f\sin\alpha$.

to coincide, and the rotation angle is denoted by ϕ measured counterclockwise to an observer looking in the direction of the positive w axis; hence the relations of (x, y) and (u, v) are given by

$$x=-R\sin\phi+u\cos\phi-v\sin\phi,$$
$$y=+R\cos\phi+u\sin\phi+v\cos\phi. \qquad (21)$$

For direction cosines of an incident ray incident on the hologram point (u, v, w), equations identical to Eqs. (9) and (10) hold.

As in the translating scanning element, on the using of Eq. (4), we have to use $\partial\phi/\partial u$ and $\partial\phi/\partial v$ instead of $\partial\phi/\partial x$ and $\partial\phi/\partial y$. In this case on account of Eqs. (21) they are related by

$$\frac{\partial\phi}{\partial u}=\frac{\partial\phi}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial\phi}{\partial y}\frac{\partial y}{\partial u}$$
$$=\frac{\partial\phi}{\partial x}\cos\phi+\frac{\partial\phi}{\partial y}\sin\phi$$
$$\frac{\partial\phi}{\partial v}=\frac{\partial\phi}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial\phi}{\partial y}\frac{\partial y}{\partial v}$$
$$=\frac{\partial\phi}{\partial x}(-\sin\phi)+\frac{\partial\phi}{\partial y}\cos\phi. \qquad (22)$$

Along the analogous procedure to the translating hologram, the parameters of the phase function for an optimum element for rotating scanning are determined so that the aberration terms may be minimized. First, since the direction cosines of the principal diffracted ray for $\phi=0$ are $(0, 0, \sin\alpha)$, we have for b_{10} and b_{01},

$$b_{10}=0,\quad b_{01}=\sin\gamma+\sin\alpha. \qquad (23)$$

The zero-th order terms are written as

$$u_0'=f(\sin\alpha+\sin\gamma-2b_{20}R)\phi-fR[3b_{30}R+(1/2)b_{11}]\phi^2,$$
$$v_0'=f\sin\alpha+b_{11}fR\phi+2b_{20}fR\phi^2. \qquad (24)$$

Since the generated curve is the straight line parallel to the u axis, we set

$$b_{11}=0. \qquad (25)$$

In v_0', the second- and higher-order terms of ϕ represent the deviation from the straightness of the generated curve.

The first-order terms of u and v are given by

$$u_1'=[(2b_{20}f\rho+f+\rho)/\rho-6b_{30}fR\phi]u+U_1(b_{20}, b_{02}, b_{21})\phi v,$$
$$v_1'=[(\rho\cos^2\alpha+f\cos^2\gamma+2b_{02}f\rho)/(\rho\cos^2\alpha)-2b_{12}fR\phi/\cos^2\alpha]v+V_1(b_{20}, b_{02}, b_{21})\phi u, \qquad (26)$$

where U_1 and V_1 are functions of b_{20}, b_{02}, and b_{21}. These expressions represent astigmatism, in which the term of $u\phi$ in u' and the term of $v\phi$ in v' are made to be zero regardless of ϕ by the choosing of coefficients as follows:

$$b_{20}=-(1/2)(1/f+1/\rho),$$
$$b_{02}=-(1/2)(\cos^2\alpha/f+\cos^2\gamma/\rho),$$
$$b_{30}=0,$$
$$b_{12}=0. \qquad (27)$$

For the remaining coefficients, due to the complexity, analytical calculations are carried out on a computer with the aid of the algebraic programming system REDUCE 3.0.[7)]

The coefficients b_{ij}, $i+j>2$ generally occur in more than one coefficient in the expressions of u' or v'. We must select the most dominant terms from them. We determine b_{21} and b_{40} from the vanish of the coefficients of $v\phi$ and $u\phi^2$ in the expression of u'. The coefficients b_{03}, b_{31}, b_{22}, b_{13}, and b_{04} are determined from the vanish of the dominant terms in the expression of v'. They are the terms of v^2,

$u\phi^2$, $v\phi^2$, uv^2, and v^3, respectively. By the vanish of these aberration terms, we obtain the remaining coefficients. We write here only the results.

$$\begin{aligned}
b_{21}&=(f^2\sin^2\gamma-f\rho\sin\alpha\sin\gamma-fR\sin\alpha\\
&\quad-\rho R\sin\alpha)/(2f^2\rho R)\\
b_{03}&=-(\rho^2\cos^2\alpha\sin\alpha+f^2\cos^2\gamma\sin\gamma)/(2f^2\rho^2)\\
b_{40}&=-(3f^3\rho\cos^2\alpha\cos^2\gamma+2f^2\rho^2\cos^2\alpha\cos^2\gamma\\
&\quad+f^2\rho^2\cos^2\alpha\sin\alpha\sin\gamma+f^2\rho R\cos^2\alpha\sin\alpha\\
&\quad-4f^2\rho R\cos^2\alpha\sin\gamma-5f\rho^2R\cos^2\alpha\sin\gamma\\
&\quad-3f^3\rho\cos^2\alpha-2f^2\rho^2\cos^2\alpha-2f^2R^2\cos^2\alpha\\
&\quad-5f\rho R^2\cos^2\alpha-3\rho^2R^2\cos^2\alpha+f^2\rho^2\cos^2\gamma\\
&\quad-f^2\rho^2\sin\alpha\sin\gamma-f^2\rho R\sin\alpha\\
&\quad-2f^2\rho R\sin\gamma-f\rho^2R\sin\gamma-f^2\rho^2\\
&\quad-f^2R^2-f\rho R^2)/(24f^3\rho^2R^2\cos^2\alpha),\\
b_{31}&=0,\\
b_{22}&=(3f\rho^2R\cos^2\alpha\sin\gamma+3f\rho R^2\cos^2\alpha\\
&\quad+3\rho^2R^2\cos^2\alpha-3f^3R\cos^2\gamma\sin\gamma\\
&\quad+2f^3\rho\cos^2\gamma-f^2\rho^2\cos^2\gamma+3f^2\rho^2\sin\alpha\sin\gamma\\
&\quad+3f^2\rho R\sin\alpha+2f^2\rho R\sin\gamma\\
&\quad-f\rho^2R\sin\gamma-2f^3\rho+f^2\rho^2+f^2R^2\\
&\quad-f\rho R^2-2\rho^2R^2)/(4f^3\rho^2R^2),\\
b_{13}&=0,\\
b_{04}&=-(5\rho^3\cos^6\alpha-15\rho^3\cos^4\alpha+9\rho^3\cos^2\alpha\\
&\quad-5f^3\cos^4\gamma+4f^3\cos^2\gamma)/(8f^3\rho^3). \qquad (28)
\end{aligned}$$

Thus, the parameters for the phase function of the rotating scanning element have been given by Eqs. (24), (26), (27), and (28).

5. Simulation by Spot Diagrams

For the estimation of the performance of our designed scanning elements, we calculate spot diagrams. The parameters are taken as the same values adopted by Herzig and Dändliker. In **Fig. 3**, we show spot diagrams at three scanning positions for the translating scanning element. In this figure, we take $\rho=50$ mm, $f=300$ mm, and $\gamma=0$, $+10.0$ deg. For incident points in the hologram, we select 10 points on each concentric circle of diameter 2.5 mm or 5.0 mm and their center for a total of 21 points. For the rotating scanning element we show spot diagrams in **Fig. 4.** In this figure, we take $\rho=60$ mm, 1000 mm, ∞, $f=430$ mm, $\alpha=0$, $+44.6$ deg, $\gamma=+44.6$ deg. Incident points are 10 points on each concentric circle of diameter 5.0/3, 10.0/3, or 5.0 mm and their center for a total of 31 points.

The calculations are carried out by the rigorous application of Eqs. (4), (6) and (7). We can

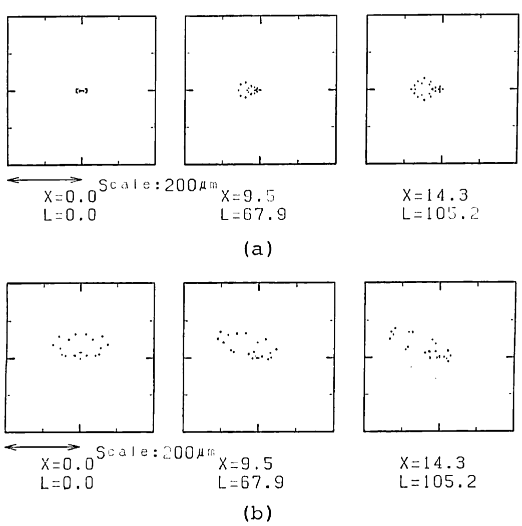

Fig. 3 Spot diagrams for the translating scanning element. $\rho=50.0$ mm, and $f=1000$ mm. L is the length of the scanned line. (a) $\gamma=0$, (b) $\gamma=+10$ deg.

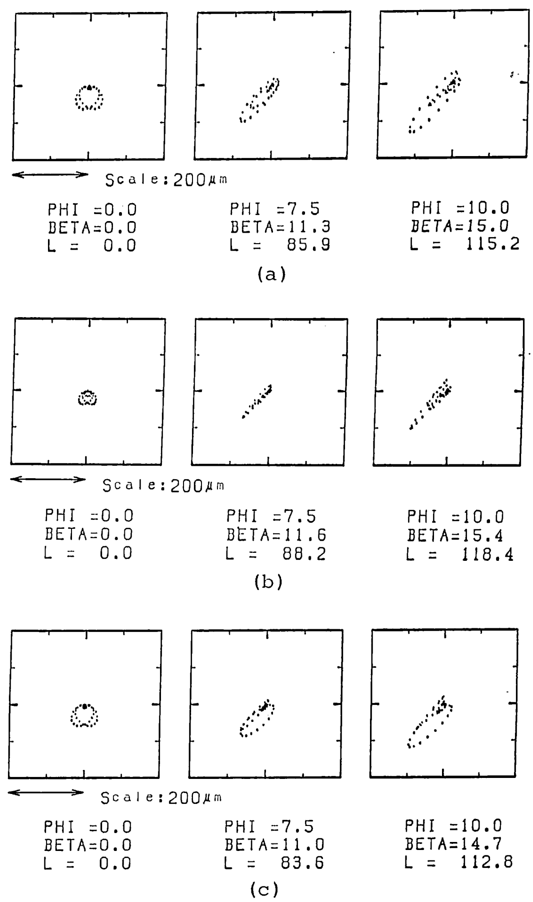

Fig. 4 Spot diagrams for the rotating element. $f=430$ mm. β is the scanned angle of the principal ray, and L is the length of the scanned line. (a) $\rho=\infty$, $\gamma=+44.6$ deg, $\alpha=44.6$ deg, (b) $\rho=-1000$ mm, $\gamma=+44.6$ deg, $\alpha=44.6$ deg, (c) $\rho=60$ mm, $\gamma=+44.6$ deg, $\alpha=0$ deg.

see a significant improvement in our spot diagrams over those of the earlier works.

6. Conclusions

While Herzig and Dändliker derived the phase function for scanning element as solutions of the differential equations, we have given it as the analytical forms. We may obtain a_{ij}, or b_{ij} to higher orders ($i+j>4$). However, such expressions of a_{ij} and b_{ij} are too complex to use practically. In our expression of u', if we determine a_{i0} and b_{i0} from the vanish of all the terms of $u\phi^i$, we have the independent part of y of the phase function of Herzig and Dändliker's first paper. Our phase functions have been supplemented by the third-and fourth-order components of y to theirs. Their second paper described two kinds of phase functions, one straight scanning and the other astigmatism-free, but, did not give the explicit expression of their phase functions. It seems that our phase function for the rotating scanner corresponds to an astigmatism-free scanner.

On the basis of the simulation using the spot diagrams, we have concluded that our design method is useful in the determination of the phase function, and that the designed scanning elements have improved performance. Our design method may be applicable to other types of holographic optical elements. We shall now prepare a paper about a holographic optical element for Fourier transform along the lines of a similar method.

Acknowledgements

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